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EIGENFUNCTIONS WITH FEW CRITICAL POINTS

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Abstract

We construct a sequence of eigenfunctions on $\mathbf{T}^2$ with a bounded number of critical points.

S. T. Yau raised a question about the number and distribution of critical points of eigenfunctions of the Laplacian on a Riemannian manifold ([4, # 76], [5, # 43]). In [6] he investigated this problem in two dimensions and proved, in particular, that under certain curvature assumptions every eigenfunction has a critical point where the critical value is uniformly bounded. Here we prove

Theorem 1. *There exists a metric on the two-dimensional torus and a sequence of eigenfunctions such that the corresponding eigenvalues go to infinity but the number of critical points remains bounded.*

This answers in the negative the question raised in [4]; however, our metric is quite special, and it is possible that for a generic metric the number of critical points increases with the growth of the eigenvalue.

The main idea of our construction is to consider a sequence of eigenfunctions $f_n(x,y) = \sin(nx+y)$ (on $\mathbf{T}^2$ with the flat metric) whose critical points lie on a union of two line segments, and then change a metric in such a way that instead of two critical "ridges" we shall have a bounded number of critical points.

We consider a Liouville metric (cf. [3])

$$q(x)\,(dx^2 + dy^2) \tag{1}$$

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on the torus $\mathbf{T}^2 = \{(x,y) \ : \ 0 \le x, y \le 2\pi\}$. Here q is a smooth periodic function whose properties we shall specify later. Joint eigenfunctions of the Laplacian $\Delta = (1/q(x))(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ and $\partial/\partial y$ have the form

$$f(x,y) \ = \ \varphi(x) \ e^{imy}, \quad m \in \mathbf{Z},$$

where φ satisfies an equation (cf. [3, (4.3)])

$$\varphi''(x) \ + \ \left(\lambda q(x) - m^2\right) \ \varphi(x) \ = \ 0.$$

In the rest of the paper we shall choose $m = 1$. Accordingly, φ satisfies

$$\varphi''(x) \ + \ (\lambda q(x) - 1) \ \varphi(x) \ = \ 0. \tag{2}$$

We choose q to be a periodic function of period $\pi/2$ and let φ satisfy (2) on $[0, \pi/2]$ with boundary conditions

$$\varphi'(0) \ = \ \varphi(\pi/2) \ = \ 0. \tag{3}$$

Then the function φ_1 defined by

$$\varphi_1(x) \ = \ \begin{cases} \varphi(x), & x \in [0, \pi/2], \\ -\varphi(\pi - x), & x \in [\pi/2, \pi], \\ -\varphi(x - \pi), & x \in [\pi, 3\pi/2], \\ \varphi(2\pi - x), & x \in [3\pi/2, 2\pi]. \end{cases} \tag{4}$$

and its shift φ_2 defined by

$$\varphi_2(x) \ = \ \varphi_1(x + \pi/2) \tag{5}$$

are two linearly independent solutions of (2) on $[0, 2\pi]$ (we are considering $x \mod 2\pi$ and using the periodicity of q).

We denote the spectrum of (2) on $[0, \pi/2]$ with boundary conditions (3) by $0 < \lambda_1 < \lambda_2 < \dots$ Then every λ_j is an eigenvalue of multiplicity two of the equation (2) on $[0, 2\pi]$ with periodic boundary conditions (the corresponding eigenfunctions $\varphi_{1,2}(j)$ are given by (4) and (5)). We next investigate the function $g_j(x)$ defined by

$$g_j(x) \ = \ \varphi_1(j)(x)^2 \ + \ \varphi_2(j)(x)^2. \tag{6}$$

Lemma 2. *There exists $C > 0$ such that for λ_j large enough the function $g_j(x)$ is monotonic outside the union of (C/λ_j)-neighborhoods of the critical points of $q(x)$.*

Proof. The solutions $\psi_j(x) = \varphi_1(j)(x) + i\,\varphi_2(j)(x)$ of the equation (2) can be asymptotically expanded in $t = \sqrt{\lambda}$ (cf. [1], [2, p. 34]). We can make a change of variable (cf. [2, p. 32])

$$\psi(x) \;=\; \exp\left\{ d_j \int_0^x \sum_{k=-1}^{\infty} t^{-k} \alpha_k(s)\, ds \right\}$$

in the equation

$$\psi'' + (t^2\, q \;-\; 1)\,\psi \;=\; 0.$$

Here $\varphi_j(0) = 1$ and

$$d_j \;=\; \frac{\psi_j'(0)}{\sum_{k=-1}^{\infty} t_j^{-k} \alpha_k(0)}$$

is the normalization constant.

Further substitution $\psi'/\psi = w$ reduces the equation above to the Ricatti equation

$$w' + w^2 + t^2 q(x) - 1 \;=\; 0$$

for $w = \sum_{k=-1}^{\infty} t^{-k} \alpha_k(x)$ from which α_k-s can be found inductively from the asymptotic expansion in t.

In particular, $\alpha_{-1}^2 + q = 0$. We assume that $q(x)$ is not identically constant and that

$$q(x) \;\gg\; 1,$$

so we can choose

$$\alpha_{-1}(x) \;=\; i\,\sqrt{q(x)}.$$

The next few terms are given by

$$
(7)\quad \begin{cases}
\alpha_0 = & -q'/(4q),\\
\alpha_1 = & (-i)\left(1/(2q^{1/2}) \;-\; 5(q')^2/(32q^{5/2}) \;+\; q''/(8q^{3/2})\right),\\
\alpha_2 = & (q''' - 4q')/(16q^2) \;-\; 9q'q''/(32q^3) \;+\; 15(q')^3/(64q^4),\\
\alpha_3 = & \dfrac{-i}{8q^{3/2}}\left(1 + \dfrac{6q'' - q''''}{4q} + \dfrac{28q'q''' + 19(q'')^2 - 50(q')^2}{16q^2}\right.\\
& \left. \qquad + \dfrac{1105(q')^4}{256q^4} - \dfrac{221(q')^2q''}{32q^3}\right),\\
\alpha_4 = & \dfrac{q(q''''' - 8q''' + 16q') + 17q''q''' + 10q'q'''' - 54q'q''}{64q^4}\\
& + \dfrac{3q'(80(q')^2 - 102(q'')^2 - 75q'q''')}{256q^5}\\
& + \dfrac{1695(q')^3(2qq'' - (q')^2)}{1024q^7}.
\end{cases}
$$

Let $h_j(x)$ be a constant multiple of the logarithmic derivative of the function $g_j(x) = \psi_j(x)\,\overline{\psi_j(x)}$,

$$h_j(x) \;=\; \frac{g_j'(x)}{2d_j g_j(x)}\;.$$

It has an asymptotic expansion in $\lambda_j = t_j^2$ given by

$$(8)\qquad h_j(x) \;=\; \sum_{k=0}^{\infty} \alpha_{2k}(x)\,\lambda_j^{-k}$$

The error term in the n-term expansion is $O(\lambda_j^{-n})$, uniformly in x and j (cf. [1], [2]). The lemma now follows from (7) and (8). q.e.d.

We next investigate the behavior of $g_j(x)$ in the C/λ_j-neighborhoods of the critical points of $q(x)$. We assume that $q(x) = q(\pi/2 - x)$ and that q has a unique minimum at 0 and a unique maximum at $\pi/4$ on $[0, \pi/2)$. The Taylor expansion of q at a critical point x_0 has the form

$$(9)\qquad q(x_0 + x) \;=\; a_0(1 + a_1x^2 + \sum_{j=2}^{\infty} a_j x^{2j}),$$

where $a_0 > 0, a_1 > 0$ at $x_0 = 0$ and $a_0 > 0, a_1 < 0$ at $x_0 = \pi/4$. It follows from the symmetries of q that $g_j(x) = g_j(-x)$, $g_j(\pi/4-x) = g_j(\pi/4+x)$.

We next differentiate (8) (cf. [1], [2]) and substitute (9) into the resulting expression to study the asymptotic expansions of $h_j'(x) = h'(x)$ (in x and $\lambda_j = \lambda$) in C/λ_j-neighborhoods of $x_0 = 0$ and $x_0 = \pi/4$. We get

$$h'(x_0 + x) \; = \; b_1(\lambda) \; + \; b_2(\lambda)x^2 \; + \; O(|x|^4 + \lambda^{-3}), \tag{10}$$

uniformly in j; here

$$\begin{cases} b_1 = \dfrac{-a_1}{2} + \dfrac{1}{\lambda a_0}\left[\dfrac{3a_2 - a_1}{2} - \dfrac{9a_1^2}{8}\right] \\ \qquad + \dfrac{1}{(\lambda a_0)^2}\left[3a_2 - \dfrac{2a_1 + 45a_3}{4} + \dfrac{a_1(324a_2 - 54a_1 - 153a_1^2)}{16}\right], \\ \dfrac{b_2}{3} = \dfrac{a_1^2}{2} - a_2 + \dfrac{1}{\lambda a_0}\left[a_1^2 - a_2 - 12a_1a_2 + \dfrac{30a_3 - 21a_1^3}{4}\right] \\ \qquad + \dfrac{1}{(\lambda a_0)^2}\left[15(a_3 - 7a_4) - a_2 + \dfrac{3a_1^2}{2} + 21a_1^3 + \dfrac{3225a_1^4}{32}\right. \\ \qquad \left. + \dfrac{3}{4}\left(122a_2^2 - 48a_1a_2 - 399a_1^2a_2 + 280a_1a_3\right)\right]. \end{cases}$$

The function $q(x)$ was chosen so that $a_1 \neq 0$ in (9). It follows that in C/λ_j-neighborhoods of the critical points

$$h_j'(x_0 + x) \; = \; \frac{-a_1}{2} \; + \; O(1/\lambda_j). \tag{11}$$

If g_j had two or more critical points in a C/λ_j-neighborhood of a critical point of q, then h_j would have at least two zeros there and h_j' would vanish, contradicting (11) for large enough λ_j. Therefore g_j has at most one critical point in every such neighborhood for large λ_j. Together with Lemma 2 this proves

Lemma 3. *The number of critical points of $g_j(x)$ is uniformly bounded above.*

We are now ready to prove the theorem. Let

$$f_j(x,y) \; = \; \varphi_1(j)(x)\sin y \; + \; \varphi_2(j)(x)\cos y, \tag{12}$$

where $\varphi_{1,2}(j)(x)$ are defined by (4) and (5). The function $f_j(x,y)$ is equal to

$$(g_j(x))^{1/2} \; \sin(\Phi_j(x) + y),$$

where $\Phi_j(x)$ is a continuous monotone function defined by

$$\cos(\Phi_j(x)) = \varphi_1(j)(x)/(g_j(x))^{1/2}, \ \sin(\Phi_j(x)) = \varphi_2(j)(x)/(g_j(x))^{1/2}$$

(Φ is monotone since a nonzero linear combination of φ_1 and φ_2 cannot have a second order zero).

At a critical point (x_0, y_0) of f_j we have

$$\frac{\partial f_j}{\partial y} \ = \ (g_j(x))^{1/2}\cos(\Phi_j(x)+y) \ = \ 0,$$

so

$$y + \Phi_j(x) = \pi/2 + \pi k, k \in \mathbf{Z}. \tag{13}$$

Also,

$$\frac{\partial f_j}{\partial x} \ = \ \frac{g_j'(x)}{2(g_j(x))^{1/2}} \ \sin(\Phi_j(x)+y) \ = \ 0$$

(we have used the equality $\cos(\Phi_j(x)+y)=0$). Accordingly, by Lemma 3, x can take a bounded number of values. Together with (13) this shows that the number of critical points of $f_j(x,y)$ is uniformly bounded above, and the proof is finished. q.e.d.

Remark. One can show that for large λ_j the eigenfunctions that were constructed have exactly 16 critical points.

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